Nonuniqueness and Turbulence

Mark A. Peterson
Physics Department, Mount Holyoke College, South Hadley MA 01075 USA
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The possibility is considered that turbulence is described by differential equations for which uniqueness fails maximally, at least in some limit. The inviscid Burgers equation, in the context of Onsager's suggestion that turbulence should be described by a negative absolute temperature, is such a limit. In this picture, the onset of turbulence coincides with the proliferation of singularities which characterizes the failure of uniqueness.

I. NONUNIQUENESS

The existence and uniqueness of solutions to the differential equations of physics is seldom an issue. The very fact that these equations describe physical reality seems to argue that their solutions must exist and be unique. Textbook examples of nonuniqueness, for example Clairaut's equation (Ref. [1], p. 94), seem like exceptional cases, which would not arise in physics in any case.

There is, however, a system of differential equations, arising as a limiting case of a much studied physical problem, in which uniqueness fails for *every* solution at *every* time. The system is singular Laplacian growth [2], [3]. One might call this behavior "maximally nonunique." It is so different from the usual behavior of differential equations that it hardly seems like a differential system at all. The way it occurs is the following: the theory of singular Laplacian growth describes the motions of certain singularities of a conformal map, which are all located on the unit circle in the complex plane. They move on the circle, but they can also "split," introducing new singularities, at any time. This behavior, which does not sound very remarkable, essentially implies the maximal nonuniqueness property. In Ref. [3] the term "fragile" was suggested for such a system, because its distinguishing feature is that its singularities can break apart.

It is natural to ask if other physical systems might reduce to a fragile system in some limit, and if the fragile property manifests itself in the behavior of the system. Turbulence is certainly a candidate to be a fragile system: like Laplacian growth, turbulence is a phenomenon in which a differential equation has unexpectedly complex solutions. In this paper I argue that turbulence is a system that has a fragile (maximally nonunique) limiting case. The limiting case is the Burgers equation. (On rather different grounds a resemblance between the problems of turbulence and Laplacian growth has been noted by Hastings and Levitov [4]).

II. INVISCID BURGERS EQUATION

The Burgers equation [5], [6]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{1}$$

has played a shifting role in the problem of turbulence. It was first introduced as a one-dimensional zero pressure version of the Navier-Stokes equation (with ν the viscosity) in the hope that it might exhibit in its solutions some of the complexity of turbulence. By the Hopf-Cole trick, though, it was found to be equivalent to a linear diffusion equation, so that the initial value problem could be solved explicitly: one can write a formula for the solution! That is surely too simple to be turbulent. Attention was thus focussed on the only case which still seemed promising, the $\nu \to 0$ limit, the inviscid Burgers equation, which is a conservation law for u. In this limit the solution may develop discontinuities ("shocks"). The distribution of shocks, and their development in time, for random initial data, is a problem called Burgers turbulence.

The reason that one should think of this as a $\nu \to 0$ limit, and not simply as a $\nu = 0$ version of the equation, is that for $\nu = 0$ the solution is ambiguous. To be sure, the solution of the inviscid equation is immediate by the method of characteristics: u is constant along lines $dx/dt = u$ (hence straight lines in the x-t plane). But if the characteristics cross, which they certainly will do in the case of random initial data $u(x, 0)$, then one has multiple determinations of $u(x, t)$, as in Fig. 1. The resolution of this difficulty is to take the formula for $u(x, t)$ in case $\nu > 0$, which is unambiguous, and let ν approach zero. The result is as shown in Fig. 2: a discontinuity forms at a definite location, separating one determination of u from another. The same procedure shows that shocks interact in a definite manner, coalescing as shown in Fig. 3 when one overtakes or collides with another.

The rules for how these discontinuities form and interact were known long ago in the theory of compressible gas dynamics, where they are called the Rankine-Hugoniot jump conditions (see Ref. [1], pp. 488-90, Ref. [6], p. 596). They are justified by appeal to the second law of thermodynamics: while u is conserved, entropy must increase in the shock. That is what the argument with $\nu > 0$ also does: we resolve the ambiguity in the inviscid Burgers equation by putting in a dissipative term with the right sign. The second law of thermodynamics says mechanical energy should be dissipated in the shock, and not created. It is this condition which leads to the interaction rule illustrated in Fig. 3.

III. NEGATIVE TEMPERATURE

Now recall the suggestion of Onsager, as emphasized by Alexandre Chorin, among others, that a statistical theory of the turbulent steady state should be characterized by a negative temperature (see Ref. [7], chapter 4). What is meant is that there is a Maxwell-Boltzmann probability distribution

$$P \sim e^{-E/kT} \tag{2}$$

which, instead of giving more weight to the low energy microstates in the ensemble description of the macrostate, as is usual, gives more weight to high energy microstates. (This statistical "temperature" of turbulence has nothing to do with usual thermodynamic temperature, which is only weakly coupled to the mechanical degrees of freedom of interest.) Let us accept this idea for the moment. The second law says that entropy should increase, or perhaps better, in this more general situation, that information should be lost, in irreversible processes. Stated in terms of the free energy

$$F = E - TS \tag{3}$$

(and remembering $T < 0$) this says that free energy should *increase* in the approach to the steady state. This is opposite to what is usual. Putting Onsager's idea together with the ideas of Burgers turbulence thus requires that mechanical energy should be created in the shock rather than dissipated. We see that the limit in the inviscid Burgers equation should be $\nu \to 0_-$, i.e., we should imagine the viscosity approaching zero through *negative* values, from *below*.

The inviscid Burger's equation does not contain temperature or ν explicitly, of course, but the rules for how the shocks form and interact are now different. The following argument gives the simple idea which is at the heart of this paper. We do not try to give it an elaborate justification, since, like many simple ideas, it may contain some truth even if the arguments are wrong. The argument is that there is a symmetry of the (general) Burgers equation,

$$\nu \to -\nu, \qquad t \to -t, \qquad x \to -x. \tag{4}$$

We can convert the $\nu < 0$ inviscid Burgers equation into the familiar $\nu > 0$ inviscid Burgers equation by reversing the signs of x and t, i.e., reflecting graphs like Fig. 3 through the origin. The result is Fig. 4, in which a single shock has spontaneously split into two, with no reference to initial conditions. This is the fragile property. If shocks can split at any time, as indicated, then the solutions to the differential equation are maximally nonunique.

IV. INTERPRETATION

Our aim in the previous section was to show that the equations which describe fluid flow, the Navier-Stokes equations, become maximally nonunique, or "fragile," in some limit. This limit is rather far removed from physical reality, however. One naturally wonders how, if at all, the nonuniqueness property might manifest itself in a real system. The example of Laplacian growth encourages one to think that real processes would not completely obscure the underlying "fragile" processes [3].

The system we have imagined is characterized by two temperatures: a "superhot" negative temperature, which describes the ensemble of microscopic entities which make up the turbulent state, and the usual temperature, which describes the ensemble at the molecular level. These two ensembles interact only weakly, it has been argued, but, again by the second law of thermodynamics, to the extent that they interact, there must be a flow of energy from the first to the second, from the turbulent ensemble to the molecular ensemble. This picture is reminiscent of the Kolmogorov cascade idea (Ref. [7], chapter 3), and suggests identifying the turbulent ensemble with the "energy range," (a hypothetical range in k space containing most of the energy), and the molecular ensemble with the "dissipation range," (a range in k space, disjoint from the energy range, in which dissipation is important), i.e., assigning a temperature gradient to the Kolmogorov picture, with negative temperature at small k and positive temperature at large k. It is in the energy range, then, at small k, that the fragile processes of nonuniqueness would occur.

These processes extract mechanical energy from the negative temperature "bath," and this energy must ultimately derive from the forces maintaining the turbulence. Thus the onset of fragile processes would appear, on the macroscopic scale, as an increase in resistance to these applied forces. At a more microscopic level it would be the onset of nonuniqueness, allowing splitting and proliferation of microscopic entities as in Fig. 4. At a still more microscopic level, corresponding to the dissipation range, the usual picture would apply, and one would have dissipative processes like Fig. 3. In this view turbulence *is* the visible manifestation of nonuniqueness on the intermediate scale of the energy range, and the work done by external forces goes directly into the proliferation of singularities at that scale, and only indirectly into dissipation.

In terms of modelling turbulence, it suggests that the proliferative processes of the energy range, which are still hypothetical, may be like the dissipative processes of the dissipation range, but time reversed (and on a larger length scale). In this way the abstract picture suggested here might persist in a more realistic dynamics.

The issue of nonuniqueness may also be relevant to CFD modelling of turbulence. The algorithms of differential equation solvers are not set up for equations

which do not have unique solutions. Experience with Laplacian growth confirms that proximity to a nonunique model may indicate trouble for conventional numerical solutions.

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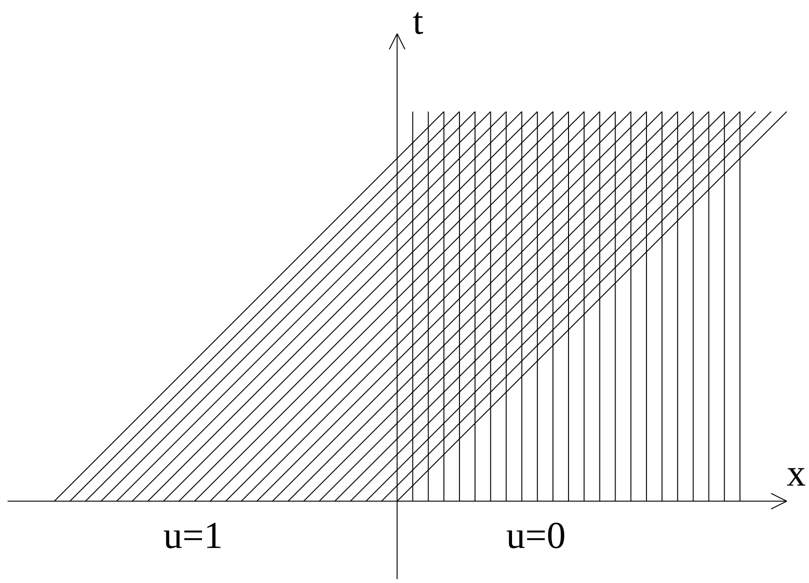

FIG. 1. The $\nu = 0$ Burgers equation with initial data $u = 1$ for $x < 0$ and $u = 0$ for $x \geq 0$ determines u(x,t) to be 1 where the characteristics have slope 1 and 0 where they are vertical. In $t \geq x \geq 0$ this determination is ambiguous.

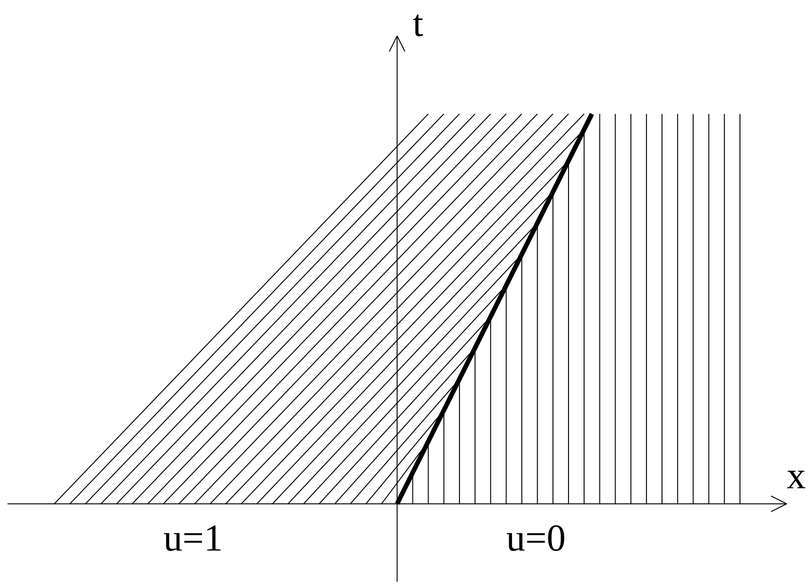

FIG. 2. The $\nu \to 0$ limit of the $\nu > 0$ Burgers equation removes the ambiguity of Fig. 1. A shock develops between the two regions.

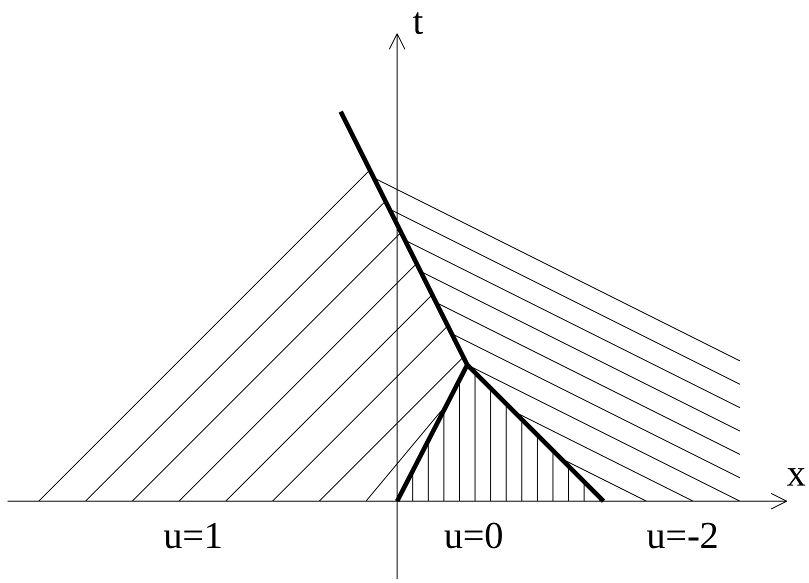

FIG. 3. The $\nu \to 0$ limit of the $\nu > 0$ Burgers equation determines how shocks interact: they coalesce, with dissipation of energy.

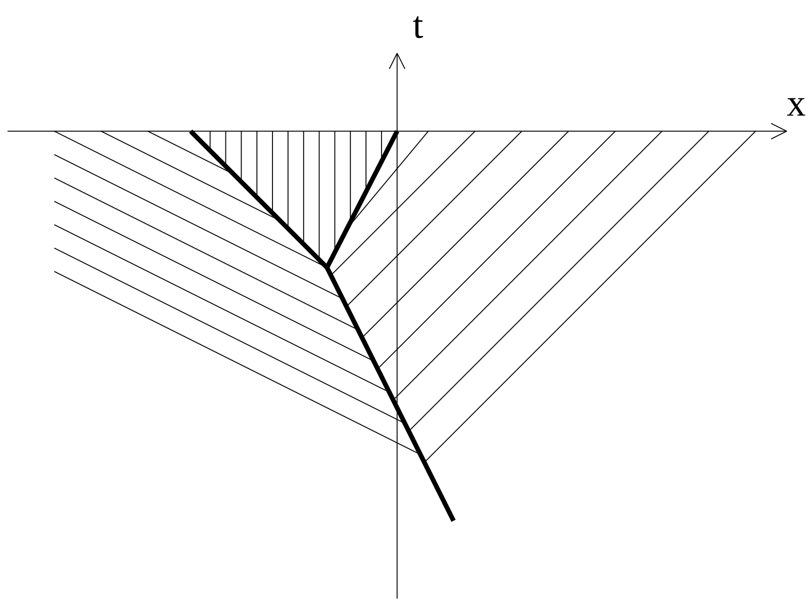

FIG. 4. It is suggested that the $\nu \to 0$ limit of the $\nu < 0$ Burgers equation should be understood as the time reversal of the usual inviscid Burgers equation. This is Fig. 3 upside-down. It shows a shock splitting at an arbitrary time. This mechanism, if it actually occurs, makes the time evolution of the Burgers equation nonunique at every time, i.e. maximally nonunique.